Announcements

1) External Reviewers Would like to talk with students 2:15 – 3 in Math Library (CB 2047) Tuesday 3/12. There will be cookies!

2) Midterm due Monday

Recall:

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Claim: f is not equal to its Taylor series.

$f(0) = 0.$

$$f'(0) = \lim_{h \to 0} \frac{e^{-1/h^2}}{h}$$

Set $h = \frac{1}{t}$ and rewrite

as two limits:

$$\lim_{t \to \infty} te^{-t^2} = \lim_{t \to \infty} \frac{t}{e^{t^2}}$$

and

$$\lim_{t \to -\infty} te^{-t^2} = \lim_{t \to \infty} \frac{t}{e^{t^2}}$$

Using l'Hopital's rule,

$$\lim_{t \to \infty} \frac{t}{e^{t^2}} = \lim_{t \to \infty} \frac{1}{2te^{t^2}}$$

$$= 0$$

Similarly,

$$\lim_{t \to -\infty} \frac{t}{e^{t^2}} = 0.$$

This shows $f'(0) = 0$.

In fact, $f^{(n)}(0) = 0$

for all $n \geq 0$.

This example is due to Cauchy and shows that

$$f(x) \neq \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

for any $x \neq 0$.

Generalizing Riemann's Integral (Section 8.1)

We know: If $f : [a,b] \to \mathbb{R}$, then if f is bounded, f is Riemann-integrable if and only if its set of discontinuities on $[a,b]$ is of Lebesgue measure zero.

Furthermore, if f is integrable, then if

$$g(x) = \int_a^x f(t)\,dt \,,$$

we have $g'(x) = f(x)$ for all $x \in [a,b]$.

Q: Is the converse true?

Namely, if

$$g'(x) = f(x) \ \forall \ x \in [a,b],$$

is it then true that

f is Riemann-integrable

and hence,

$$g(x) = \int_a^x f(t)\,dt \ ?$$

Nasty counterexample

The function on the cover of this book!

Let $g(x) = \begin{cases} x^2 \sin(1/x), & x > 0 \\ 0, & x \leq 0 \end{cases}$

g' is discontinuous at

$x = 0$

Why?

If $x \neq 0$,

$$g'(x) = 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$$

If $x = 0$,

$$g'(0) = \lim_{h \to 0} \frac{h^2 \sin\left(\frac{1}{h}\right)}{h}$$

$$= \lim_{h \to 0} h\sin\left(\frac{1}{h}\right)$$

$= 0$ by the Squeeze Theorem

$$\lim_{x \to 0} g'(x)$$

$$= \lim_{x \to 0} \left(2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) \right)$$

which does not exist.

Therefore, g' is discontinuous at zero.

The Idea: Use g to construct a function f such that f' is discontinuous on the Cantor set.

But the Cantor set is of measure zero!

Instead, modify the Cantor construction by, at the n^{th} stage, removing intervals of length $\frac{1}{3^{n+1}}$ instead of $\frac{1}{3^n}$. The intersection S of all such sets does not have Lebesgue measure zero.

The Fix: The generalized Riemann integral.

Definition: (tagged partition)

A partition P of $[a,b]$ is said to be **tagged** if for each interval $[x_i, x_{i+1}]$ in P, we choose a point c_i in this interval. Write as

$$(P, \{c_i\}_{i=1}^{n})$$

if P has n intervals.

Definition: (gauge)

A gauge on an interval $[a,b]$ is just a positive real-valued function on $[a,b]$

Definition: (gauged partition)

Let $\delta : [a,b] \to \mathbb{R}$ be a gauge. A tagged partition $(P, \{c_i\}_{i=0}^{n-1})$ is $\delta(x)$-fine if

$$\boxed{x_i - x_{i-1} < \delta(c_i)}$$

$\forall\ i,\ 0 \leq i \leq n-1.$

Theorem: (Riemann integrability)

$f: [a,b] \to \mathbb{R}$ is Riemann integrable iff there is a number A such that for every $\varepsilon > 0$ $\exists$ $\delta > 0$ such that for all δ-fine partitions $(P, \{c_i\}_{i=0}^{n-1})$,

$$\left| A - \sum_{i=0}^{n-1} f(c_i)(x_{i+1} - x_i) \right| < \varepsilon$$

Thankfully, Amanda has already pretty much done this argument in class, so we will skip.

Definition: (generalized Riemann Integral)

$f : [a,b] \to \mathbb{R}$ has **generalized Riemann integral** A if for every $\varepsilon > 0$, $\exists$ a gauge $\delta : [a,b] \to \mathbb{R}$ such that for all $\delta(x)$-fine partitions $(P, \{c_i\}_{i=0}^{n-1})$,

$$\left| A - \sum_{i=0}^{n-1} f(c_i)(x_{i+1} - x_i) \right| < \varepsilon.$$

Theorem: (Fundamental!)

Suppose $g: [a,b] \to \mathbb{R}$ and $f = g'$. Then f has a generalized Riemann integral and, denoting the integral abusively by

$$\int_a^b f(x)\,dx,$$

$$\boxed{g(x) - g(a) = \int_a^x f(t)\,dt}$$

Proof: Observe that,

for $x \in [a,b]$,

$$g(x) - g(a) = \sum_{i=0}^{n-1} \left(g(x_{i+1}) - g(x_i) \right)$$

for any partition P of

$[a,x]$.

Now, for simplicity, let $x = b$ and let $(P, \{c_i\}_{i=0}^{n-1})$ be a tagged partition on $[a,b]$.

Then

$$\left| g(b)-g(a) - \sum_{i=0}^{n-1} f(c_i)(x_{i+1}-x_i) \right|$$

$$= \left| \sum_{i=0}^{n-1} \Big(g(x_{i+1}) - g(x_i) - f(c_i)(x_{i+1}-x_i) \Big) \right|$$

$$\leq \sum_{i=0}^{n-1} \underbrace{\left| g(x_{i+1}) - g(x_i) - f(c_i)(x_{i+1}-x_i) \right|}_{\text{estimate this quantity}}$$